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Orbital Equations in a Weak Gravitational Field

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With the obtainment of the Synge equations of motion in third approximation for a test particle, we determine the relativistic force in the field generated by a massive body obtained previously in terms of its potentials. Afterwards, in order to study the motion, we obtain the integral of energy and the integral of angular momentum. By means of them, the general form for the trajectory of an equatorial orbit and for the advance of its apsidal line are obtained. As we will seȩ, the known diverse contributions for this advance appears in the general form after strong supplementary conditions on the potentials. As application, with such assumptions these contributions are derived in a unified way.

1. INTRODUCTION

In an earlier paper (Gambi, 1983), Synge's approximation method has been applied to obtain the weak gravitational field of a massive body with an axis of symmetry around which it is rotating steadily (Synge, 1970). The method was carried out to include the second approximations, which is enough to obtain third-order equations of motion. This means that terms of order m^2 are retained as significant and that there is an error of order m^3 in the field equations, m being the mass of the body.

The result is more sophisticated than the one obtained in the standard post-Newtonian approximation (Chandrasekhar, 1965; Weinberg, 1972) because the $O(m^2)$ terms in the second-order deviations $\underset{2}{\gamma_{\mu\nu}}$ [(6) below] are explicitly determined and the same happens with the $\overset{2}{O}(m^{5/2})$ terms of $\underset{2}{\gamma_{\mu 4}}$.

The purpose of the present work is to use the whole metric obtained in Gambi (1983) in order to derive the contributions of these terms to the motion of a small body or test particle.

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In order to this, in Section 2 the results of Gambi (1983) relevant to the present work are briefly summarized. Next, in Section 3 we obtain first the orbital equations in third approximation for any stationary weak gravitational field and afterwards the corresponding ones to the particular case which we are considering. Next, taking into account that, owing to the characteristics of the field, there are two first integrals of motion, namely, the energy one and the one of angular momentum, in Section 4 we get the first one and in Section 5 we get the second one. These integrals depend not only on the classical Newtonian potential and on the potential of rotation but also on the stress of the body. Using them, supposing that the generating body has an equatorial plane of symmetry, in Section 6 the general equation for the trajectory of an equatorial orbit and the value for the advance of its perihelion are obtained. The known advances are particular contributions of this advance. Assuming that the deviation of the massive body with respect to sphericity is small (in a technical sense) and that the potentials due to stress and rotation are almost inversely proportional to the distance, these contributions can then be obtained. As application of the general form we derive them jointly.

2. THE GRAVITATIONAL FIELD

The weak gravitational field which we are considering is generated by a massive body rotating steadily around its axis of symmetry.

Assuming as topology of space-time the one of a Euclidean 4-space, taking coordinates $x_a (a = 1, 2, 3, 4)$ in such a way that $x_\mu (\mu = 1, 2, 3)$ are rectangular Cartesian coordinates and $x_4 = it$, and choosing as axis of symmetry the Ox_3 axis, we have

$$u_1 = -\frac{u}{r} x_2, \qquad u_2 = \frac{u}{r} x_1, \qquad u_3 = 0, \qquad u_4 = i, \tag{1}$$

$$\frac{\partial u_\alpha}{\partial t} = 0, \qquad \frac{\partial \rho}{\partial t} = 0 \tag{2}$$

$$u_\alpha = O(\kappa^{1/2}), \qquad \rho = O(\kappa) \tag{3}$$

where $u_\alpha (\alpha = 1, 2, 3)$ and u are, respectively, the Eulerian 3-velocity and the velocity of the body satisfying $u = u(r, x_3)(r^2 = x_1^2 + x_2^2)$, and $\rho = \rho(r, x_3)$ is its Eulerian density. κ is the small dimensionless parameter of the same order of the mass of the body which constitutes the basis of the approximation. All magnitudes are measured in seconds.

Denoting by I the history of the world tube (supposed simply connected) corresponding to the body and by E the part of space-time exterior to I, if the energy tensor [which is given by

$$\begin{aligned} T^{\mu\nu} = \rho u_\mu u_\nu - S_{\mu\nu}, \qquad T^{\mu 4} = i\rho u_\mu, \qquad T^{44} = -\rho < 0, \qquad &\text{in } I \\ T^{ab} = 0, \qquad \text{in } E \end{aligned} \tag{4}$$

where $S_{\mu\nu} = O(\kappa^2)$ is the Eulerian stress of the body] is such that the Eulerian equations of motion are almost satisfied, i.e., if

$$\begin{aligned} \rho\frac{du_\mu}{dt} - S_{\mu\nu,\nu} &= \rho V_{,\mu} + O(\kappa^3) \\ \frac{d\rho}{dt} + \rho u_{\mu,\mu} &= O(\kappa^{7/2}) \end{aligned} \tag{5}$$

then the metric deviations of second order $\underset{2}{\gamma_{ab}}$ $(a, b = 1, 2, 3, 4)$ with respect to the Minkowskian metric δ_{ab} are given by

$$\begin{aligned} \underset{2}{\gamma_{\mu\nu}} &= 2(V + V^2)\delta_{\mu\nu} - 2\varkappa J[(\rho u_\mu u_\nu)^*] \\ &\quad + 4J[\tfrac{1}{2}\varkappa S^*_{\mu\nu} + (V^2)_{,\mu\nu} - V_{,\mu}V_{,\nu}] + O(\kappa^3) \\ \underset{2}{\gamma_{\mu 4}} &= -2\varkappa iJ(\rho u_\mu) - 8iJ(V_{,\sigma}W_{\sigma,\mu} - V_{,\mu\sigma}W_\sigma \\ &\quad + W_\mu \Delta V - V\Delta W_\mu) + O(\kappa^{7/2}) \\ \underset{2}{\gamma_{44}} &= -2(V - V^2) - \varkappa JS_{\sigma\sigma} + \varkappa J(\rho u_\sigma u_\sigma) + O(\kappa^3) \end{aligned} \tag{6}$$

where

$$\begin{aligned} J[f(\mathbf{x}, t)] &= -(4\pi)^{-1}\int f(\mathbf{x}', t)|\mathbf{x} - \mathbf{x}'|^{-1}\, d_3x' \\ S^*_{\mu\nu} = S_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}S_{\sigma\sigma}, &\qquad (\rho u_\mu u_\nu)^* = (\rho u_\mu u_\nu) - \tfrac{1}{2}\delta_{\mu\nu}(\rho u)^2 \\ \varkappa &= 8\pi \end{aligned} \tag{7}$$

V is the Newtonian potential

$$V = -4\pi J(\rho) \tag{8}$$

and

$$W_\mu = -4\pi J(\rho u_\mu) \tag{9}$$

Thus we have a universe which contains a body of arbitrary shape with the only restriction being that it possess an axis of symmetry around which it is rotating with small velocity, $u_\mu = O(\kappa^{1/2})$, in its own gravitational field; the density ρ is small, $O(\kappa)$. The energy tensor is given by (4), where $S_{\mu\nu}$

is the Newtonian stress under gravity. $S_{\mu\nu}$ satisfies (5) and is $O(\kappa^2)$. The metric tensor is

$$g_{ab} = \delta_{ab} + \underset{2}{\gamma_{ab}} \tag{10}$$

where $\underset{2}{\gamma_{ab}}$ is given by (6).

The distant field corresponding to this universe is given by

$$\begin{aligned} &g_{\mu\nu} = \delta_{\mu\nu} + 2\delta_{\mu\nu}(m+m')r^{-1}, \qquad g_{44} = 1 - 2(m+m')r^{-1} \\ &g_{14} = 2ix_2 r^{-3} J_3, \qquad g_{24} = -2ix_1 r^{-3} J_3, \qquad g_{34} = 0 \end{aligned} \tag{11}$$

where m and J_3 are the mass and the angular momentum (with respect to Ox_3) of the body and $m' = \int \rho V d_3 x'$ is the mass of the field.

3. THE ORBITAL EQUATIONS

The fact that gravitational fields are shown geometrically leads in a natural way to establish the geodesic hypothesis as the basic law of motion for test particles. In this way its motion is determined only by the potentials of the massive body or bodies which generate the field. So, it is possible to speak about the substratum metric produced by these bodies and is in this metric where the geodesic hypothesis is applied.

To begin with all generality, let us make no initial assumption about the form of the metric g_{ab}. Accordingly with the geodesic hypothesis, orbits satisfy the equations

$$\ddot{x}_\rho + \underset{2}{\Gamma}{}^{\rho}_{mn} \dot{x}_m \dot{x}_r = \theta \dot{x}_\rho, \qquad \underset{2}{\Gamma}{}^{4}_{mn} \dot{x}_m \dot{x}_r = i\theta \tag{12}$$

where $\underset{2}{\Gamma}{}^{l}_{mn}$ are the Christoffel symbols of second kind, θ is a Lagrange multiplier, and

$$\dot{x}_\rho = u_\rho = O(\kappa^{1/2}) \tag{13}$$

The first three equations (12) are equivalent to

$$\dot{u}_\rho + \underset{2}{\Gamma}{}^{\rho}_{\mu\nu} u_\mu u_\nu + 2i \underset{2}{\Gamma}{}^{\rho}_{\mu 4} u_\mu - \underset{2}{\Gamma}{}^{\rho}_{44} = \theta u_\rho \tag{14}$$

Now, if the field is stationary, by a straightforward calculation the Christoffel symbols become

$$\begin{aligned} &\underset{2}{\Gamma}{}^{\rho}_{\mu\nu} = \tfrac{1}{2} g^{\rho\alpha}[g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}] + \tfrac{1}{2} g^{\rho 4}[g_{\mu 4,\mu} + g_{\nu 4,\mu}] \\ &\underset{2}{\Gamma}{}^{\rho}_{\mu 4} = \tfrac{1}{2} g^{\rho\alpha}[g_{4\alpha,\mu} - g_{\mu 4,\alpha}] + \tfrac{1}{2} g^{\rho 4} g_{44,\mu} \\ &\underset{2}{\Gamma}{}^{\rho}_{44} = -\tfrac{1}{2} g^{\rho\alpha} g_{44,\alpha} \end{aligned} \tag{15}$$

and substituting (15) in (14) we get

$$\dot{u}_\rho+\{\tfrac{1}{2}g^{\rho\alpha}[g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}-g_{\mu\nu,\alpha}]+\tfrac{1}{2}g^{\rho4}[g_{\mu4,4}+g_{\nu4,\mu}]\}u_\mu u_\nu$$
$$+2i\{\tfrac{1}{2}g^{\rho\alpha}(g_{4\alpha,\mu}-g_{4\mu,\alpha})+\tfrac{1}{2}g^{\rho4}g_{44,\mu}\}u_\mu+\tfrac{1}{2}g^{\rho\alpha}g_{44,\alpha}=\theta u_\rho \tag{16}$$

If we now suppose that the metric g_{ab} has deviations $\underset{2}{\gamma}_{ab}=O(\kappa^2)$ these equations are reduced to

$$\dot{u}_\rho+[\tfrac{1}{2}(\underset{2}{\gamma}_{\mu\rho,\nu}+\underset{2}{\gamma}_{\nu\rho,\mu}+\underset{2}{\gamma}_{\mu\nu,\rho}-\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{\mu\alpha,u}-\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{\nu\alpha,\mu}$$
$$+\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{\mu\nu,\alpha})-\tfrac{1}{2}(\underset{2}{\gamma}_{4\rho}\underset{2}{\gamma}_{4\mu,\nu}+\underset{2}{\gamma}_{4\rho}\underset{2}{\gamma}_{4\nu,\mu})]u_\mu u_\nu$$
$$+i(\underset{2}{\gamma}_{4\rho,\mu}-\underset{2}{\gamma}_{4\mu,\rho}-\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{4\alpha,\mu}+\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{4\mu,\alpha}-\underset{2}{\gamma}_{\rho4}\underset{2}{\gamma}_{44,\mu})u_\mu$$
$$+\tfrac{1}{2}(\underset{2}{\gamma}_{44,\rho}-\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{44,\alpha})=\theta u_\rho \tag{17}$$

Now, in order to obtain equations of motion in third approximation we eliminate only the terms that, with all surety, are $O(\kappa^3)$. Thus equations (17) are reduced to

$$\dot{u}_\rho+\tfrac{1}{2}\underset{2}{\gamma}_{44,\rho}-\tfrac{1}{2}\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{44,\alpha}+\tfrac{1}{2}(\underset{2}{\gamma}_{\mu\rho,\nu}+\underset{2}{\gamma}_{\nu\rho,\mu}-\underset{2}{\gamma}_{\mu\nu,\rho})u_\mu u_\nu$$
$$+i(\underset{2}{\gamma}_{4\rho,\mu}-\underset{2}{\gamma}_{4\mu,\rho})u_\mu=\theta u_\rho \tag{18}$$

Back now to the fourth equation (12), we get for his corresponding Christoffel symbols

$$\underset{2}{\Gamma}{}^4_{\mu\nu}=\tfrac{1}{2}g^{4\rho}[g_{\mu\rho,\nu}+g_{\nu\rho,\mu}-g_{\mu\nu,\rho}]+\tfrac{1}{2}g^{44}(g_{4\mu,\nu}+g_{4\nu,\mu})$$
$$\underset{2}{\Gamma}{}^4_{\mu4}=\tfrac{1}{2}g^{4\alpha}[g_{4\alpha,\mu}-g_{4\mu,\alpha}]+\tfrac{1}{2}g^{44}g_{44,\mu} \tag{19}$$
$$\underset{2}{\Gamma}{}^4_{44}=-\tfrac{1}{2}g^{4\alpha}g_{44,\alpha}$$

with that on substituting in it, it results

$$\{\tfrac{1}{2}g^{4\rho}[g_{\mu\rho,\nu}+g_{\nu\rho,\mu}-g_{\mu\nu,\rho}]+\tfrac{1}{2}g^{44}[g_{4\mu,\nu}+g_{4\nu,\mu}]\}u_\mu u_\nu$$
$$+2i[\tfrac{1}{2}g^{4\alpha}(g_{4\alpha,\mu}-g_{4\mu,\alpha})+\tfrac{1}{2}g^{44}g_{44,\mu}]u_\mu$$
$$+\tfrac{1}{2}g^{4\rho}g_{44,\rho}=i\theta \tag{20}$$

Introducing now the metric deviations, we have

$$\begin{aligned}
&[\tfrac{1}{2}(-\underset{2}{\gamma}_{4\rho})(\underset{2}{\gamma}_{\mu\rho,\nu}+\underset{2}{\gamma}_{\nu\rho,\mu}-\underset{2}{\gamma}_{\mu\nu,\rho})\\
&\quad+\tfrac{1}{2}(1-\underset{2}{\gamma}_{44})(\underset{2}{\gamma}_{4\mu,\nu}+\underset{2}{\gamma}_{4\nu,\mu})]u_\mu u_\nu\\
&\quad+2i[\tfrac{1}{2}(-\underset{2}{\gamma}_{4\alpha})(\underset{2}{\gamma}_{4\alpha,\mu}-\underset{2}{\gamma}_{4\mu,\alpha})+\tfrac{1}{2}(1-\underset{2}{\gamma}_{44})\underset{2}{\gamma}_{44,\mu}]u_\mu\\
&\quad+\tfrac{1}{2}(-\underset{2}{\gamma}_{4\rho})(\underset{2}{\gamma}_{44,\rho})=i\theta
\end{aligned}\tag{21}$$

and, on taking away the terms which surely are $O(\kappa^3)$, it becomes

$$i\theta=\underset{2}{\gamma}_{\mu 4,\nu}u_\mu u_\nu+i\underset{2}{\gamma}_{44,\mu}(u_\mu-\underset{2}{\gamma}_{44}u_\mu)-\tfrac{1}{2}\underset{2}{\gamma}_{\rho 4}\underset{2}{\gamma}_{44,\rho}\tag{22}$$

Now, multiplying (22) by iu_β and simplifying, it results in

$$\theta u_\beta=\underset{2}{\gamma}_{44,\mu}u_\mu u_\beta+O(\kappa^3)\tag{23}$$

and carrying this equation to (18) we obtain

$$\begin{aligned}
&\dot{u}_\rho+\tfrac{1}{2}\underset{2}{\gamma}_{44,\rho}-\tfrac{1}{2}\underset{2}{\gamma}_{\rho\alpha}\underset{2}{\gamma}_{44,\alpha}+\tfrac{1}{2}(\underset{2}{\gamma}_{\mu\rho,\nu}+\underset{2}{\gamma}_{\nu\rho,\mu}-\underset{2}{\gamma}_{\mu\nu,\rho})u_\mu u_\nu\\
&\quad+i(\underset{2}{\gamma}_{4\rho,\mu}-\underset{2}{\gamma}_{4\mu,\rho})u_\mu=\underset{2}{\gamma}_{44,\mu}u_\mu u_\rho+O(\kappa^3)
\end{aligned}\tag{24}$$

The final form of the equations depends of the field in which the test particle is moving. If one wishes to study the motion at great distance, it is enough to consider only the field (11), but this involves the missing in the motion description of some elements which characterize the generating body [as seen in (11) the stress tensor has not survived to describe the field]. This is the reason that is better to use the whole metric (10). Carrying then (10) to (24) and neglecting $O(\kappa^3)$ terms, it results in

$$\begin{aligned}
&\dot{u}_\rho-V_{,\rho}+2VV_{,\rho}-\tfrac{1}{2}\varkappa(JS_{\sigma\sigma})_{,\rho}+\tfrac{1}{2}\varkappa[J(\rho u_\sigma u_\sigma)]_{,\rho}\\
&\quad+\tfrac{1}{2}[(2V\delta_{\mu\rho})_{,\nu}+(2V\delta_{\nu\rho})_{,\mu}-(2V\delta_{\mu\nu})_{,\rho}]u_\mu u_\nu\\
&\quad+i\{[-2\varkappa iJ(\rho u_\rho)]_{,\mu}+[2\varkappa iJ(\rho u_\mu)]_{,\rho}\}u_\mu=-2V_{,\mu}u_\mu u_\rho+O(\kappa^3)
\end{aligned}\tag{25}$$

and from here, taking into account (9), we have

$$\begin{aligned}
\dot{u}_\rho=V_{,\rho}&+V_{,\rho}(u^2-4V)-4u_\rho\dot{V}+P_{,\rho}-Q_{,\rho}\\
&+4(W_{\rho,\mu}-W_{\mu,\rho})u_\mu+O(\kappa^3)
\end{aligned}\tag{26}$$

where

$$P=\tfrac{1}{2}\varkappa JS_{\sigma\sigma}=-\int S_{\sigma\sigma}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{27}$$

and

$$Q=\tfrac{1}{2}\varkappa J(\rho u_\sigma u_\sigma)=-\int \rho u_\sigma u_\sigma|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{28}$$

P is therefore the stress-trace potential of the body and is the one due to its rotation.

Writing equations (26) in the form

$$\dot{u}_\rho = V_{,\rho}+F_\rho \tag{29}$$

we see in the first term the Newtonian force per unit mass. The rest

$$\begin{aligned}F_\rho = {} & V_{,\rho}(u^2-4V)-4u_\rho\dot{V}+P_{,\rho}-Q_{,\rho}\\ & +4(W_{\rho,\mu}-W_{\mu,\rho})u_\mu+O(\varkappa^3)=O(\kappa^2)\end{aligned} \tag{30}$$

is the relativistic perturbing force per unit mass. This force not only depends on the classical potential V, on the test particle velocity, and even on the body's rotation, but also on its own body stress.

(26) is Synge's third-order equation of motion for a test particle. That this is true can be verified by imposing supplementary conditions on his equations [see (1.61), (1.62), and (1.63) of Synge, 1970]: (i) Consider only Synge's equations for the motion of two bodies; (ii) one of the bodies is very small with respect to the other; (iii) this second body is later in steady rotation. By imposing the conditions one can ignore the self-potentials and stress in the small body and vanish the terms in which derivatives with respect to t appear in the other body. Then doing this we obtain in a straightforward way equations (26) but at the same time, which is important, we verify the geodesic hypothesis on his equations for the case which we are considering. The generality with which the field has been obtained has been what has allowed, in short, to get the equations (26) of which, on the other hand, the traditional ones, which correspond to the field (11), are a particular case.

4. THE INTEGRAL OF ENERGY

Accordingly with what was previously said, in the part of space-time exterior to the world tube of the body, orbits satisfy the equations (12) or, what is the same, the Lagrangian equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\mu}-\frac{\partial L}{\partial x_\mu}=0 \tag{31}$$

where

$$L=(-g_{\mu\nu}\dot{x}_\mu\dot{x}_\nu-2ig_{\mu4}\dot{x}_\mu+g_{44})^{1/2} \tag{32}$$

Now, by the conditions (2), these equations have the first integral

$$L-\dot{x}_\mu\frac{\partial L}{\partial\dot{x}_\mu}=1+E \tag{33}$$

where the constant E is the total energy per unit mass of the test particle.
From (33) it results in

$$L^{-1}\left(L^2-\frac{1}{2}\dot{x}_\mu\frac{\partial L^2}{\partial\dot{x}_\mu}\right)-1=E \tag{34}$$

and by (32), it becomes

$$L^{-1}(-ig_{\mu4}\dot{x}_\mu+g_{44})-1=E \tag{35}$$

Then carrying (10) to (35) we have

$$E=\tfrac{1}{2}(u^2+\underset{2}{\gamma_{44}})+\tfrac{1}{2}\underset{2}{\gamma_{\mu\nu}}u_\mu u_\nu+\tfrac{1}{8}(u^2-\underset{2}{\gamma_{44}})(3u^2+\underset{2}{\gamma_{44}})+O(\kappa^3) \tag{36}$$

where as before $u^2=u_\mu u_\mu$.
Substituting now in (36) the deviations (6) and taking into account that

$$\underset{2}{\gamma_{\mu\nu}}=O(\kappa),\qquad \underset{2}{\gamma_{\mu4}}=O(\kappa^{3/2}),\qquad \underset{2}{\gamma_{44}}=O(\kappa),\qquad \dot{x}_\mu=u_\mu=O(\kappa^{1/2}) \tag{37}$$

it results in

$$\begin{aligned}(\tfrac{1}{2}u^2-V)-\tfrac{1}{2}\varkappa JS_{\sigma\sigma}+\tfrac{1}{2}\varkappa J(\rho u^2)&\\ +\tfrac{3}{8}u^4+\tfrac{3}{2}u^2V+\tfrac{1}{2}V^2+O(\kappa^3)&=E\end{aligned} \tag{38}$$

As it is seen, the principal part of (38) is the Newtonian total energy per unit mass, as the corresponding potential energy is $-V$, not V. According to (36), (3), and (8), this part is $O(\kappa)$ and the rest of the left-hand side of (38) is $O(\kappa^2)$. So, retaining only the principal part in (38), we have

$$\tfrac{1}{2}u^2-V+O(\kappa^2)=E \tag{39}$$

Then, applying (39), the integral of energy can be written in the following way:

$$(\tfrac{1}{2}u^2-V)-\tfrac{1}{2}\varkappa JS_{\sigma\sigma}+\tfrac{1}{2}\varkappa J(\rho u^2)+5V^2+6EV=E-\tfrac{3}{2}E^2+O(\kappa^3) \tag{40}$$

in it directly only the magnitudes which characterize the space-time considered.

5. THE INTEGRAL OF ANGULAR MOMENTUM

When a test particle moves in the gravitational field of a body in stationary motion, generally all that can be assured is the existence of the integral of energy discussed in the preceding section. But if the body has an axis of symmetry, the azimuthal angle ϕ is a cyclic coordinate and the angular momentum is kept, i.e.,

$$\frac{\partial L}{\partial \dot{\phi}} = -A \tag{41}$$

where the constant A is the angular momentum per unit mass of the test particle.

Going to Cartesian coordinates x_μ (in such a way, as was said in Section 2, that the axis of symmetry is Ox_3), (41) is reduced to

$$x_1 \frac{\partial L}{\partial \dot{x}_2} - x_2 \frac{\partial L}{\partial \dot{x}_1} = -A \tag{42}$$

or to

$$-\frac{1}{2} L^{-1}\left(x_1 \frac{\partial L^2}{\partial \dot{x}_2} - x_2 \frac{\partial L^2}{\partial \dot{x}_1} \right) = A \tag{43}$$

and from here, to

$$L^{-1}[x_1 g_{2\mu} \dot{x}_\mu - x_2 g_{1\mu} \dot{x}_\mu + i(x_1 g_{24} - x_2 g_{14})] = A \tag{44}$$

Imposing now the conditions of weakness and of slow motion we have as before (37). Then (44) is reduced to

$$\begin{aligned} &(1 + \tfrac{1}{2}u^2 - \tfrac{1}{2}\underset{2}{\gamma_{44}})(x_1 u_2 - x_2 u_1) + (x_1 \underset{2}{\gamma_{2\mu}} u_\mu - x_2 \underset{2}{\gamma_{1\mu}} u_\mu) \\ &\quad + i(x_1 \underset{2}{\gamma_{24}} - x_2 \underset{2}{\gamma_{14}}) + O(\kappa^{5/2}) = A \end{aligned} \tag{45}$$

and substituting (6) in (45), we have

$$(1 + \tfrac{1}{2}u^2 + 3V)(x_1 u_2 - x_2 u_1) - 4(x_1 w_2 - x_2 w_1) + O(\kappa^{5/2}) = A \tag{46}$$

On the other hand, as

$$u^2 = 2(E + V) + O(\kappa^2) \tag{47}$$

accordingly with (38), then substituting (47) in (46) it results

$$(1 + E + 4V)(x_1 u_2 - x_2 u_1) - 4(x_1 w_2 - x_2 w_1) + O(\kappa^{5/2}) = A \tag{48}$$

In order to obtain a similar expression to the well-known Newtonian one, we make the transformation to cylindrical coordinates

$$x_1 = R\cos\phi, \qquad x_2 = R\sin\phi \tag{49}$$

In this way (48) is reduced to

$$(1+E+4V)R^2\dot\phi - 4(x_1W_2 - x_2W_1) + O(\kappa^{5/2}) = A \tag{50}$$

But as

$$AE = ER^2\dot\phi + O(\kappa^{5/2}) \tag{51}$$

it results

$$(1+4V)R^2\dot\phi - 4(x_1W_2 - x_2W_1) + O(\kappa^{5/2}) = A(1-E) \tag{52}$$

and finally

$$R^2\dot\phi = 4(x_1W_2 - x_2W_1) + h[1 - 4V + O(\kappa^2)] \tag{53}$$

where $h = A(1-E)$.

6. THE TRAJECTORY OF AN EQUATORIAL ORBIT

As we have already see, the orbital equations (26) have both first integrals (40) and (53), Ox_3 being the axis of the massive body.

Let us now assume that the body has not only an axis of symmetry but also an equatorial plane of symmetry ($x_3 = 0$). Then these two integrals are enough to determine an equatorial orbit. So, putting $r = R$, the expressions (40) and (53) are written in polar coordinates as follows:

$$\dot r^2 + r^2\dot\phi^2 = 2(E+V) + 2P - 2Q - 10V^2 - 12EV - 3E^2 + O(\kappa^3) \tag{54}$$

$$r^2\dot\phi = h(1-4V) + 4(x_1W_2 - x_2W_1) + O(\kappa^{5/2}) \tag{55}$$

Then, doing the traditional change $\xi = 1/r$ and taking into account that $h = O(\kappa^{1/2})$, we have from (55)

$$\begin{aligned}\dot r^2 + r^2\dot\phi^2 = \left[\left(\frac{d\xi}{d\phi}\right)^2 + \frac{1}{r^2}\right]&[h^2(1-4V)^2 \\ &+ 8h(1-4V)(x_1W_2 - x_2W_1) + 16(x_1W_2 - x_2W_1)^2]\end{aligned} \tag{56}$$

and carrying this expression to (54), it results

$$\begin{aligned}\left(\frac{d\xi}{d\phi}\right)^2 + \frac{1}{r^2} = h^{-2}&[2(E+V) + 2P - 2Q + 6V^2 + 4EV - 3E^2 \\ &- h^{-3}[16(E+V)(x_1W_2 - x_2W_1)] + O(\kappa^2)\end{aligned} \tag{57}$$

or definitively

$$\left(\frac{d\xi}{d\phi}\right)^2 = -F(\xi) \tag{58}$$

where

$$F(\xi) = \xi^2 - h^{-2}[2(E+V)+2P-2Q+6V^2+4EV-3E^2] + h^{-3}[16(E+V)(x_1W_2-x_2W_1)]+O(\kappa^2) \tag{59}$$

(58) is the equation of the wanted trajectory. Let us now study the advance of its apsidal line.

As it is known, the inverse apsidal distances ξ_1, ξ_2 satisfy

$$F(\xi_1) = F(\xi_2) = 0, \qquad \xi_2 > \xi_1 > 0 \tag{60}$$

the apsidal angle is

$$\Delta\phi = \int_{\xi_1}^{\xi_2} [-F(\xi)]^{-1/2}\, d\xi \tag{61}$$

and the advance of perihelion per revolution is

$$\varepsilon = 2\Delta\phi - 2\pi \tag{62}$$

In order to calculate the integral (61) we write $F(\xi)$ in the form

$$F(\xi) = \xi^2 + a\xi + b - B(\xi) \tag{63}$$

with

$$a = -2mh^{-2}, \qquad b = -2Eh^{-2} > 0 \tag{64}$$

where $m = \int \rho\, d_3x$ is the mass of the massive body.

Then we have

$$B(\xi) = h^{-2}[2U+2P-2Q+6V^2+4EV-3E^2 - h^{-1}16(E+V)(x_1W_2-x_2W_1)]+O(\kappa^2) \tag{65}$$

where

$$U = V - m\xi \tag{66}$$

Next, taking into account (60), we get

$$\xi_1^2 + a\xi_1 + b = B(\xi_1) = B_1, \qquad \xi_2^2 + a\xi_2 + b = B(\xi_2) = B_2 \tag{67}$$

and hence

$$\begin{aligned} a + \xi_1 + \xi_2 &= (B_2 - B_2)/(\xi_2 - \xi_2) \\ b - \xi_1\xi_2 &= (\xi_1 B_2 - \xi_2 B_1)/(\xi_1 - \xi_2) \end{aligned} \tag{68}$$

Finally, since F vanishes for $\xi=\xi_1$ and $\xi=\xi_2$ we take out the factor $(\xi-\xi_1)(\xi-\xi_2)$ and, eliminating a and b, we write $F(\xi)$ in the form

$$F(\xi)=(\xi-\xi_1)(\xi-\xi_2)[1-G(\xi)] \tag{69}$$

where

$$G(\xi)=\frac{(B-B_1)(\xi-\xi_2)-(B-B_2)(\xi-\xi_1)}{(\xi-\xi_1)(\xi-\xi_2)(\xi_1-\xi_2)} \tag{70}$$

As can be seen, the function G has no singularities in the ends of the interval (ξ_1, ξ_2). So we can write the integral (61) in the following form:

$$\Delta\phi=\int_{\xi_1}^{\xi_2}[(\xi_2-\xi)(\xi-\xi_1)]^{-1/2}[1-G(\xi)]^{-1/2}\,d\xi \tag{71}$$

This is the general expression for the advance of perihelion of an equatorial orbit in the field considered. Now, in order to obtain actual results, let us add supplementary assumptions.

First we assume the generating body is nearly spherical, that is to say, that the deviation with respect to sphericity is $O(\kappa)$. In this case we may expand V in the usual form

$$V=m\xi+\mu_3\xi^3+\mu_5\xi^5+\cdots \tag{72}$$

where μ_3 is the quadripole potential and $\mu_5,\ldots$ are potentials of higher order. Second, we assume that the potentials due to stress and rotation are, in the equatorial plane, almost inversely proportional to the distance, i.e.,

$$P=p\xi+O(\kappa^3), \qquad Q=q\xi+O(\kappa^3) \tag{73}$$

where $p=\text{constant}=O(\kappa^2)$ and $q=\text{constant}=O(\kappa^2)$. Then, accordingly with (72), we have

$$U=\mu_3\xi^3+\mu_5\xi^5+\cdots=O(\kappa^2) \tag{74}$$

Carrying now (73) and (74) to (65), it results in

$$\begin{aligned}B(\xi)=h^{-2}(&2\mu_3\xi^3+2\mu_5\xi^5+2p\xi-2q\xi+6m^2\xi^2\\&+4Em\xi-3E^2-h^{-1}8EJ_3\xi^2-h^{-1}8mJ_3\xi^3)+O(\kappa^2)\end{aligned} \tag{75}$$

and hence we have

$$B(\xi)=b_0+b_1\xi+b_2\xi^2+b_3\xi^3+\cdots+O(\kappa^2) \tag{76}$$

where

$$
\begin{aligned}
b_0 &= -3E^2h^{-2} \\
b_1 &= 2h^{-2}(p-q+2mE) \\
b_2 &= 6m^2h^{-2}-8EJ_3h^{-3} \\
b_3 &= 2\mu_3h^{-2}-8mJ_3h^{-3} \\
&\vdots
\end{aligned}
\tag{77}
$$

Now, as the coefficients (77) are $O(\kappa)$, we have that B and G are $O(\kappa)$. Thus, expanding G by the binomial theorem and taking into account (62) we see that the advance of perihelion is

$$\varepsilon = \int_{\xi_1}^{\xi_2} [(\xi_2-\xi)(\xi-\xi_1)]^{-1/2} G(\xi)\,d\xi + O(\kappa^2) \tag{78}$$

Since G is linear in B, from (78) we deduce

$$\varepsilon = \varepsilon_0 b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3 + \cdots + O(\kappa^2) \tag{79}$$

where

$$\varepsilon_n = \int_{\xi_1}^{\xi_2} [(\xi_2-\xi)(\xi-\xi_1)]^{-1/2} \frac{(\xi^n-\xi_1^n)(\xi-\xi_2)-(\xi^n-\xi_2^n)(\xi-\xi_1)}{(\xi-\xi_1)(\xi-\xi_2)(\xi_1-\xi_2)}\,d\xi \tag{80}$$

The result of these integrals for $n=0$, 1, 2 and 3 is

$$\varepsilon_0 = 0, \qquad \varepsilon_1 = 0, \qquad \varepsilon_2 = \pi, \qquad \varepsilon_3 = 3\pi m h^{-2} + O(\kappa) \tag{81}$$

So, carrying these values to (73) and taking into account (77), we have finally

$$\varepsilon = 6\pi m^2h^{-2} - 8\pi EJ_3h^{-3} + 6\pi m\mu_3h^{-4} - 24\pi m^2J_3h^{-5} + \cdots + O(\kappa^2) \tag{82}$$

In this global expression we see in the first term the relativistic advance for the Schwarzschild field. The second and fourth terms are the relativistic advances due to the rotation of the massive body and the third term is the Newtonian advance due to its oblateness. The remaining terms, not evaluated explicitly in (82), are Newtonian advances due to higher moments. As it is seen, under the stated assumptions, all terms shown in (82) are $O(\kappa)$, i.e., of the same order of the mass of the massive body.

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